JACOBI MATRICES AND QUADRATURE RULES ON THE UNIT CIRCLE AND THE REAL LINE

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AIM OF THE TALK

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta) \approx \sum_{j=1}^{n} \lambda_j g(z_j) = I_n(g)$$

$$\lambda_j > 0,\ |z_j| = 1,\ j = 1, \dots, n \text{ and } z_j \neq z_k \text{ if } j \neq k.$$

- I. PRELIMINARY RESULTS.
- II. SYMMETRIC WEIGHT FUNCTIONS.
- III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES.

I. PRELIMINARY RESULTS

General Problem

Approximate calculation of integrals:

$$I_\sigma(f) = \int_a^b f(x)dx \approx \sum_{j=1}^{n} A_j f(x_j) = I_n^\sigma(f)$$

$\{x_j\}_{j=1}^n \subset [a,b]$, $x_j \neq x_k$ if $j \neq k$.

Gauss Formulas

Set $\mathbb{P}_k = \text{span}\{x^j : j = 0,1,\dots,k\}$, $\mathbb{P} = \bigcup \mathbb{P}_k$

$I_n^\sigma(f) = \sum_{j=1}^{n} A_j f(x_j)$ is the n-point Gauss formula to

$I_\sigma(f) = \int_a^b f(x)\sigma(x)dx \Leftrightarrow I_\sigma(P) = I_n(P)$, $\forall p \in \mathbb{P}_{2n-1}$.

Orthogonal Polynomials

$\{Q_k\}_{k=0}^{\infty}$ be the sequence of orthonormal polynomials w.r.t. $\sigma(x)$ on $[a,b]$. Then,

$I_n^\sigma(f) = \sum_{j=1}^{n} A_j f(x_j)$ is the n-point Gauss formula to $I_\sigma(f) \Leftrightarrow$

1. $\{x_j\}_{j=1}^{n}$ are the zeros of $Q_n(x)$.
2. $A_j = \left[\sum_{k=0}^{n-1} |Q_k(x_j)|^2 \right]^{-1}$, $j = 1, \ldots, n$ (Christoffel numbers)

Features:

1. Optimality,
2. Positivity,
3. Convergence,
4. Efficient computation.

Jacobi Matrices

$\{Q_k\}_{k=0}^{\infty}$: orthonormal sequence. It holds

$$xQ_n(x) = a_{n+1}Q_{n+1}(x) + b_nQ_n(x) + a_nQ_{n-1}(x),\ n \geq 0,\ Q_{-1} = 0.$$

$$\text{Setting, } \mathcal{J} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

then, the eigenvalues of the n-th truncation of the matrix $\mathcal{J}$ are the nodes of $I_n^{\sigma}(f)$ and the square of the first component of the eigenvector of unit length corresponding to the eigenvalue x_j yields the weight A_j, $j = 1, \ldots, n$.

Periodic Integrands

$$I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta,$$

where both g and ω are 2π-periodic functions and ω is a weight function on any interval of length 2π.

Example

$$G(a,b) = \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \ (\omega(\theta) \equiv 1), \ \text{(Elliptic integral)}$$

Estimation of $G(a,b)$: Gauss-Legendre vs. Trapezoidal rule.

a	b	Gauss-Legendre	Trapezoidal
2.0	3.0	$4.16233E-09$	$4.43762E-12$
0.3	0.2	$4.16233E-08$	$4.43762E-11$
4.0	5.0	$2.11498E-11$	$6.20000E-17$
1.0	2.0	$6.40799E-07$	$7.55989E-08$
0.9	1.0	$4.77908E-12$	$1.00000E-17$
0.4	0.8	$1.60200E-06$	$1.88997E-07$
0.3	0.6	$2.13600E-06$	$2.51996E-07$
0.9	1.6	$1.83510E-07$	$3.86458E-09$
0.9	1.8	$7.11999E-07$	$8.39988E-08$

I. PRELIMINARY RESULTS

$$I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta \approx I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(\theta_j),$$

$\{\theta_j\}_{j=1}^n \subset [-\pi, \pi)$, $\theta_j \neq \theta_k$ if $j \neq k$.

Szegő Formula

$I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(\theta_j)$ is an n-point Szegő formula to $I_\omega(g) \Leftrightarrow I_\omega(T) = I_n^\omega(T)$ for any trigonometric polynomial of degree $\leq n-1$. By setting $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$, $z_j = e^{i\theta}$, $j = 1, \ldots, n$ and taking into account that $T(\theta) = \sum_{k=0}^{n-1}(a_k \cos k\theta + b_k \sin k\theta) = \sum_{k=-(n-1)}^{n-1} A_k z^k$, $(z = e^{i\theta})$. Then, $I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(z_j)$ is an n-point Szegő formula for $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta \Leftrightarrow I_n^\omega(L) = I_\omega(L)$ for any $L \in \Lambda_{-(n-1),n-1} = \text{span}\{z^k : -(n-1) \leq k \leq n-1\}$ (Laurent Polynomials).

Szegő polynomials

$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z : |z| < 1\}$, $p, q \in \mathbb{Z}$, $p \leq q$
$\Lambda_{p,q} = \text{span}\{z^k : p \leq k \leq q\}$, $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$.
$\{\rho_k\}_{k=0}^{\infty}$: the sequence of monic Szegő polynomials for $\omega(\theta)$ i.e. for each $k \geq 0$, $\rho_k(z) = z^k + \cdots$ and for $n \geq 1$ $\langle \rho_n, z^k \rangle_\omega = \int_{-\pi}^{\pi} \rho_n(z)\overline{z^k}\omega(\theta)d\theta = 0$, $k = 0, 1, \ldots, n-1$. (The zeros of ρ_n lie in $\mathbb{D}$).
Then, $I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ with $z_j \in \mathbb{T}$ and $z_j \neq z_k$ if $j \neq k$ is an n-point Szegő formula to $I_\omega(g) \Leftrightarrow$

1. $\{z_j\}_{j=1}^{n}$ are the zeros of $B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$ with $\tau_n \in \mathbb{T}$ and $\rho_{n-1}^*(z) = z^{n-1}\overline{\rho_{n-1}(1/\overline{z})}$.
2. $\lambda_j = \left[\sum_{k=0}^{n-1} \frac{|\rho_k(z_j)|^2}{\|\rho_k\|_\omega^2}\right]^{-1} > 0$, $j = 1, \ldots, n$, $\|\rho_k\|_\omega^2 = \langle \rho_k, \rho_k \rangle_\omega$.

Example

$$\omega(\theta) \equiv 1, \quad \rho_n(z) = z^n, \quad B_n(z, \tau_n) = z^n + \tau_n.$$

Take: $\tau_n = (-1)^{n+1}$, $z_j = e^{i\theta_j}$, $\theta_j = -\pi + \dfrac{2\pi j}{n}$, $j = 1, \ldots, n$

$\lambda_j = \dfrac{2\pi}{n}$, $j = 1, \ldots, n.$

$$\int_{-\pi}^{\pi} g(\theta) d\theta \approx \sum_{j=1}^{n} \lambda_j g(\theta_j) = \frac{2\pi}{n} \sum_{j=1}^{n} g\left(-\pi + \frac{2\pi j}{n}\right).$$

(Trapezoidal rule)

Hessemberg Matrices

$\delta_n = \rho_n(0)$, $n \geq 0$, $\delta_0 = 1$, $|\delta_n| < 1$, $n \geq 1$. (Verblunsky Parameters). $\rho_n(z)$: n-th monic Szegő polynomial. For $\tau \in \mathbb{T}$, set

$$H_n(\tau, \delta_0, \ldots, \delta_{n-1}) = H_n(\tau) = D_n^{-1/2} \begin{pmatrix} -\delta_1 & -\delta_2 & \cdots & -\delta_{n-1} & -\tau \\ \sigma_1^2 & -\overline{\delta_1}\delta_2 & \cdots & -\overline{\delta_1}\delta_{n-1} & -\overline{\delta_1}\tau \\ 0 & \sigma_2^2 & \cdots & \overline{\delta_2}\delta_{n-1} & -\overline{\delta_2}\tau \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1}^2 & -\overline{\delta_{n-1}}\tau \end{pmatrix} D_n^{1/2},$$

$\sigma_k = \sqrt{1 - |\delta_k|^2}$, $D_n = \mathrm{diag}[\gamma_0, \ldots, \gamma_{n-1}] \in \mathbb{R}^{n \times n}$ $\gamma_0 = 1$ and $\gamma_k = \gamma_{k-1}\sigma_k^2 > 0$, $k = 1, \ldots, n-1$.

Then,

$H_n(\tau)$ is an unreduced unitary upper Hessenberg matrix so that its eigenvalues are the nodes of an n-point Szegő formula for $\tilde{\tau} = \frac{\tau + \delta_n}{1 + \tau\overline{\delta_n}}$ and the square of the first component of the eigenvector of unit length associated with z_j yields the weight λ_j, $j = 1, \ldots, n$.

Prescribed nodes on $\mathbb{T}$: Szegő-type formulas

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta \approx \sum_{j=1}^{n} \lambda_j g(z_j) = I_n^\omega(g).$$

$\{z_j\}_{j=1}^n$: the zeros of $B_n(z,\tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$, $|\tau_n| = 1$.

- Fix $z_\alpha \in \mathbb{T}$, take $\tilde{\tau}_n = -\frac{z_\alpha \rho_{n-1}(z_\alpha)}{\rho_{n-1}^*(z_\alpha)} \in \mathbb{T} \Rightarrow$ the corresponding n-point Szegő formula has a prescribed node at z_α (Szegő-Radau).
- Fix $z_\alpha, z_\beta \in \mathbb{T}$, $n > 2$, there exist $\tilde{\delta}_{n-1} \in \mathbb{D}$ and $\tilde{\tau}_n \in \mathbb{T}$ so that it holds,

1. $\tilde{B}_n(z,\tilde{\tau}_n) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n \tilde{\rho}_{n-1}^*(z)$ where $\tilde{\rho}_{n-1}(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho_{n-2}^*(z)$ has n distinct zeros on $\mathbb{T}$: $z_\alpha, z_\beta, z_1 \dots, z_{n-2}$.
2. There exist positive numbers A, B and λ_j, $j = 1, \dots, n-2$:

$$\tilde{I}_n^\omega(g) = Ag(z_\alpha) + Bg(z_\beta) + \sum_{j=1}^{n-2} \lambda_j g(z_j) = I_\omega(g),\ g \in \Lambda_{-(n-2),n-2}$$

(Szegő-Lobatto)

II. SYMMETRIC WEIGHT FUNCTIONS

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta, \quad \omega(-\theta) = \omega(\theta): \text{ for any } \theta \in [-\pi, \pi]$$

$I_n^\omega(g) = \displaystyle\sum_{j=1}^{n} \lambda_j g(z_j)$, $(\{z_j\} \subset \mathbb{T})$ is said to be *symmetric* if the nodes are real or appear on $\mathbb{T}$ in complex conjugate pairs.

Theorem

Let ω be a symmetric weight function and $I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(z_j)$ an n-point Szegő formula generated by $B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$, $\tau_n \in \mathbb{T}$. Then,

1. *$I_n^\omega(g)$ is symmetric $\Leftrightarrow \tau_n \in \{\pm 1\}$.*
2. *Suppose $z_j = \overline{z}_k$ for some j and k: $1 \leq j, k \leq n$, then*

$$\lambda_j = \lambda_k.$$

Computation of nodes and weights reduces one half.

Theorem

Let ω be a symmetric weight function and fix z_α and $\overline{z}_\alpha$ on $\mathbb{T}$. Then, for $n > 2$ there exist positive weights A and λ_j, $j = 1, \ldots, n-2$ and distinct nodes $z_j \in \mathbb{T} \setminus \{z_\alpha, \overline{z}_\alpha\}$, $j = 1, \ldots, n-2$ uniquely determined so that setting

$$\tilde{I}_n^\omega(g) = A\left[g(z_\alpha) + g(\overline{z}_\alpha)\right] + \sum_{j=1}^{n-2} \lambda_j g(z_j),$$

the following holds,

1. $\tilde{I}_n^\omega(g)$ *is symmetric.*
2. $\tilde{I}_n^\omega = I_\omega(g)$, $\forall g \in \Lambda_{-(n-2),(n-2)}$.

Again,

Computation of nodes and weights reduces one half.

II. SYMMETRIC WEIGHT FUNCTIONS

Hessemberg Matrices

$$H_n(\tau, \delta_0, \ldots, \delta_{n-1}) = H_n(\tau) = D_n^{-1/2} \begin{pmatrix} -\delta_1 & -\overline{\delta_2} & \cdots & -\overline{\delta_{n-1}} & -\overline{\tau} \\ \sigma_1^2 & -\delta_1\overline{\delta_2} & \cdots & -\delta_1\overline{\delta_{n-1}} & -\delta_1\overline{\tau} \\ 0 & \sigma_2^2 & \cdots & \delta_2\overline{\delta_{n-1}} & -\delta_2\overline{\tau} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1}^2 & -\delta_{n-1}\overline{\tau} \end{pmatrix} D_n^{1/2},$$

Jacobi Matrices

$$\mathcal{J}_m = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & b_2 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{m-1} & b_{m-1} \end{pmatrix}, \quad m \approx E\left[\frac{n}{2}\right]$$

$\{a_k\}_1^\infty$, $\{b_k\}_0^\infty$ given in terms of $\{\delta_k\}_0^\infty$.

II. SYMMETRIC WEIGHT FUNCTIONS

Proposition

$\omega(\theta)$ is a symmetric weight function on $[-\pi, \pi]$, if and only if, there exists a weight function σ on $[-1, 1]$ such that

$$\omega(\theta) = \sigma(\cos\theta)|\sin\theta|.$$

Furthermore, it holds

$$\int_{-1}^{+1} f(x)\sigma(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta,$$

$$g(e^{i\theta}) = f\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right).$$

The connection between the unit circle and the interval $[-1,1]$

Take $r, s \in \{0,1\}$,

$$I^{\sigma}_{n,(r,s)} = rA_+f(1) + sA_-f(-1) + \sum_{j=1}^{n-r-s} A_j f(x_j),\ \{x_j\} \subset (-1,1)$$

$x_j = \cos\theta_j$, $\theta_j \in (0,\pi)$, A_+, A_-, A_j real.
Set $z_j = e^{i\theta_j} \in \mathbb{T}$ and define

$$I^{\omega}_{2n-r-s} = 2[rA_+g(1) + sA_-g(-1)] + \sum_{j=1}^{2(n-r-s)} A_j[g(z_j) + g(\overline{z}_j)]$$

Then

$$\begin{aligned} I^{\sigma}_{n,(r,s)}(f) &= I_\sigma(f) \text{ for all } f \in \mathbb{P}_N \Leftrightarrow \\ I^{\omega}_{2n-r-s}(g) &= I_\omega(g) \text{ for all } g \in \Lambda_{-N,N}. \end{aligned}$$

II. SYMMETRIC WEIGHT FUNCTIONS

To fix ideas, take $r = s = 0$ and consider $I_n^\sigma(f) = \sum_{j=1}^n A_j f(x_j)$ the n-point Gauss formula to $I_\sigma(f) = \int_{-1}^{+1} f(x)\sigma(x)dx$.
Then: $I_n^\sigma(f) = I_\sigma(f), \forall f \in \mathbb{P}_{2n-1}$, $N = 2n - 1$.
Setting $x_j = \cos\theta_j$, $\theta_j \in (0, \pi)$, $z_j = e^{i\theta_j}$ $j = 1, \ldots, n$. Then

$$I_{2n}^\omega(g) = \sum_{j=1}^{n} A_j[g(z_j) + g(\overline{z}_j)] = I_\omega(g), \ \forall g \in \Lambda_{-(2n-1),(2n-1)}$$

($2n$-point Symmetric Szegő formula)

- $r = 1$, $s = 0$ or $r = 0$, $s = 1 \Rightarrow$ Gauss-Radau to $I_\sigma(f)$
- $r = s = 1 \Rightarrow$ Gauss-Lobatto.

Conclusion

Computation of symmetric Szegő formulas for $\omega(\theta)$ leads to the computation of Gauss, Gauss-Radau, and Gauss-Lobatto formulas for $\sigma(x)$ such that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta| \Rightarrow$ Jacobi Matrices.

II. SYMMETRIC WEIGHT FUNCTIONS

$\omega(\theta)$: symmetric on $[-\pi, \pi]$, available information $\{\delta_k\}_0^\infty$

$$\delta_k = \rho_k(0),\ k = 0, 1, \dots\ \delta_0 = 1,\ \delta_k \in (-1, 1),\ k = 1, 2, \dots$$

There exists $\sigma(x)$ on $[-1, 1]$: $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$.
The Jacobi matrix for $\sigma(s)$ is required!

$$\mathcal{J} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},\ \{a_k\}_1^\infty, \{b_k\}_0^\infty?$$

Theorem (Geronimus relations)

Let ω be a symmetric weight function on $[-\pi, \pi]$ and σ the weight function on $[-1, 1]$ related to ω by

$$\omega(\theta) = \sigma(\cos\theta)|\sin\theta|.$$

Let $\{\delta_k\}_{k=0}^{\infty}$ be the sequence of Verblunsky parameters for ω and $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the coefficients of the Jacobi matrix for σ. Then, the following holds:

$$2a_n = \sqrt{(1-\delta_{2n})(1-\delta_{2n-1}^2)(1+\delta_{2n-2})}, \;\; n \geq 1,$$

$$2b_n = \delta_{2n-1}(1-\delta_{2n}) - \delta_{2n+1}(1+\delta_{2n}), \;\; n \geq 0.$$

An experiment involving Rogers-Szegő polynomials

$\omega(\theta)$ symmetric on $[-\pi,\pi] \Leftrightarrow \omega(\theta)=\sigma(\cos\theta)|\sin\theta|$, $\sigma(x)$ being a weight function on [-1,1].
Consider the *wrapped* Gaussian measure

$$\omega(\theta)=\omega(q;\theta)=\frac{1}{\sqrt{2\pi\log(1/q)}}\sum_{j=-\infty}^{+\infty}\exp\left(\frac{-(\theta-2\pi j)^2}{2\log(1/q)}\right),$$

$0<q<1$, *Rogers-Szegő polynomials*.

- Trigonometric moments: $\mu_k=\int_{-\pi}^{\pi}e^{-ik\theta}\omega(\theta)d\theta=q^{\frac{k^2}{2}}$, $k\geq 0$.
- Verblunsky parameters: $\delta_k=\rho_k(0)=(-1)^k q^{\frac{k}{2}}$, $k\geq 0$.
- Jacobi coefficients for $\sigma(x)$:

$$\begin{aligned} a_n &= \frac{1}{2}\sqrt{(1-q^n)(1-q^{2n-1})(1+q^{n-1})}, \ \ n\geq 1, \\ b_n &= \frac{1}{2}q^{n-\frac{1}{2}}\left(q^{n+1}+q^n+q-1\right), \ n\geq 0. \end{aligned}$$

II. SYMMETRIC WEIGHT FUNCTIONS

Computation of n-point symmetric rules

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta, \quad I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(z_j).$$

- Hessemberg Matrix: $H_n(\tau_n, \delta_0, \ldots, \delta_{n-1})$, $\tau_n \in \{\pm 1\}$. Suppose n even, $\tau_n = 1$, $q = 0.2$.
- Jacobi Matrix of dimension $\frac{n}{2}$.

Computational Time

n	$Jacobi$	$Hessenberg$
100	0.016	0.0234
200	0.047	2.969
300	0.172	11.281
400	0.454	30.45
500	0.937	81.828
600	1.73	172

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

Estimation of the integral $I_\sigma(f) = \int_{-1}^{+1} f(x)\sigma(x)dx$.

Gauss Formula $I_n^\sigma(P) = \sum_{j=1}^{n} A_j P(x_j) = I_\sigma(P), \forall P \in \mathbb{P}_{2n-1}$

- $\{x_j\}_{j=1}^n$ the zeros of any orthogonal polynomial of degree n with respect to $\sigma(x)$.
- No freedom is left to fix some nodes in advance.
- Fix nodes at $\pm 1 \Rightarrow$ Gauss-Radau and Gauss-Lobatto formulas.
- Further step: fix a node $x_\alpha \in (-1, 1)$.

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

Problem

Set $I_\sigma = \int_{-1}^{+1} f(x)\sigma(x)dx$. Then, given $x_\alpha \in (-1,1)$, $r, s, \in \{0,1\}$ and $n > 1+r+s$ find positive weights A_+, A_-, A_α and $\{A_j\}_{j=1}^{n-r-s-1}$ along with distinct nodes $\{x_j\}_{j=1}^{n-r-s-1} \subset (-1,1)$ such that

$$
\begin{aligned}
I_n^{r,s}(f) &= rA_+f(1) + sA_-f(1) + A_\alpha f(x_\alpha) + \sum_{j=1}^{n-r-s-1} A_j f(x_j) \\
&= I_\sigma(f), \text{ for all } f \in \mathbb{P}_{2(n-1)-r-s}
\end{aligned}
$$

$\dim(\mathbb{P}_{2(n-1)-r-s}) = 2n - (r+s+1) =$ number of free parameters in $I_n^{r,s}(f)$.

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

Recent Contributions

- **S. González-Pinto, D. Hernández-Abreu, J. I. Montijano.** *An efficient family of strongly A-stable Runge-Kutta collocation methods for stiff system and DAEs. Part I. Stability and order results.* J. Compt. Appl. Math. (2010) To appear.
- **A. Bultheel, R. Cruz-Barroso and M. Van Barel** *On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line.* Calcolo (2010). To appear.

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

Solution: by passing to the unit circle.
$\sigma(x)$, $x \in [-1,1] \Leftrightarrow \omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ (Symmetric). Recall,
Given $n > 2$, z_α and $\overline{z}_\alpha \in \mathbb{T}$ there exist $n-2$ distinct nodes $\{z_j\}_{j=1}^{n-2} \subset \mathbb{T}\backslash\{z_\alpha, \overline{z}_\alpha\}$ and positive weights A_α and λ_j, $j = 1, \ldots, n-2$ so that

$$\tilde{I}_n^\omega(g) = A_\alpha[g(z_\alpha) + g(\overline{z}_\alpha)] + \sum_{j=1}^{n-2} \lambda_j g(z_j)$$

(n-point Szegő-Lobatto symmetric formula), holds,

1) $\tilde{I}_n^\omega(g)$ is symmetric, 2) $\tilde{I}_n^\omega(g) = I_\omega(g)$, $\forall g \in \Lambda_{-(n-2),(n-2)}$
Furthermore, the nodes $z_\alpha, \overline{z}_\alpha$ and z_j, $j = 1, \ldots, n-2$ are the zeros of $\tilde{B}_n(z, \tilde{\tau}_n) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n \rho_{n-1}^*(z)$ with $\tilde{\rho}_{n-1}(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho_{n-2}^*(z)$ where $\tilde{\delta}_{n-1} \in (-1,1)$, $\tilde{\tau}_n \in \{\pm 1\}$ are uniquely determined and easily computable, $\rho_{n-2}(z)$ the $(n-2)$-th monic Szegő polynomial for $\omega(\theta)$.

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

Theorem

Given r and s in $\{0,1\}$ and $x_\alpha \in (-1,1)$, take $z_\alpha \in \mathbb{T}$ such that $x_\alpha = \Re(z_\alpha)$. Let $\sigma(x)$ be a weight function on $[-1,1]$ and define $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\theta \in [-\pi,\pi]$. For $n > 1 + r + s$ consider the $(2n-r-s)$-point Szegő-Lobatto symmetric formula for $I_\omega(g)$ with prescribed nodes at z_α and $\overline{z}_\alpha$. Let $\tilde{\delta}_{2n-(r+s-1)} \in (-1,1)$ and $2n - \tilde{(}r+s) \in \{\pm 1\}$ be the parameters characterizing this formula. Then the quadrature rule,

$$I_n^{r,s}(f) = rA_+ f(1) + sA_- f(-1) + A_\alpha f(x_\alpha) + \sum_{j=1}^{n-r-s-1} A_j f(x_j)$$

satisfying $I_n^{r,s}(f) = I_\sigma(f)$, $\forall f \in \mathbb{P}_{2(n-1)-r-s}$ there exists and is unique, if and only if,

$$\tilde{\tau}_{2n-(r+s)} = (-1)^\gamma$$

Furthermore, the weights of $I_n^{r,s}(f)$ are positive.

III. COMPUTATION OF GAUSS FORMULAS WITH PREASSIGNED NODES

The parameter $\tilde{\delta}_{2n-(r+s+1)}$ and $\tau_{2n-(r+s)}$ are computed in terms of the Szegő polynomials for $\omega(\theta)$.

Thus, the Verblunsky parameters $\{\delta_k\}_0^{2n-(r+s+2)}$ are required.

But the initial available information is concerned with $\sigma(x)$:
$\{a_k\}_1^\infty$ and $\{b_k\}_0^\infty$ (Jacobi coefficients).

Then, how to obtain the sequence $\{\delta_k\}_0^\infty$?

$$\text{Set } \tilde{\mathcal{J}} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ a_1^2 & b_1 & 1 & 0 & \cdots \\ 0 & a_2^2 & b_2 & 1 & \cdots \\ 0 & 0 & a_3^2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{(monic Jacobi matrix).}$$

Consider the unique LU decomposition of $\tilde{\mathcal{J}} + I$ (I identity matrix) i.e.

$$\tilde{\mathcal{J}} + I = \mathcal{L}\mathcal{U}.$$

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where

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ u_2 & 1 & 0 & 0 & \cdots \\ 0 & u_4 & 1 & 0 & \cdots \\ 0 & 0 & u_6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_3 & 1 & 0 & \cdots \\ 0 & 0 & u_5 & 1 & \cdots \\ 0 & 0 & 0 & u_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Once the sequence $\{u_k\}_{k=1}^{\infty}$ has been computed, then it holds,

$$\delta_k = 1 - \frac{2u_k}{1+\delta_{k-1}},\ k \geq 1,\ \delta_0 = 1.$$

Numerical examples involving Berstein-Szegő polynomials

Assume $\sigma(x) = \frac{(1-x)^\alpha (1+x)^\beta}{P(x)}$, $\alpha, \beta \in \{\pm\frac{1}{2}\}$, $P(x) > 0$ on [-1,1].

Now, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta| = \frac{(1-\cos\theta)^{\alpha+\frac{1}{2}}(1+\cos\theta)^{b+\frac{1}{2}}}{|h(z)|^2}$, $z = e^{i\theta}$
with $h(z)$ an algebraic polynomial of the same degree as $P(x)$.
To fix ideas, $\alpha = \beta = -\frac{1}{2}$ (Chebyshev-type of the first kind).
$P(x) = (1 - x\gamma)^m$, $\gamma \in (-1, 1)$, $m \geq 0$ i.e.

$$\sigma(x) = \frac{1}{(1 - x\gamma)^m \sqrt{1 - x^2}}, \; x \in [-1, 1]$$

$$\omega(\theta) = \sigma(\cos\theta)|\sin\theta| = \frac{1}{|z - \tilde{\gamma}|^{2m}}, \; z = e^{i\theta}, \; \tilde{\gamma} \in (-1, 1).$$

The monic Szegő polynomials $\rho_n(z)$ are explicitly known

$$\rho_n(z) = z^{n-m}(z - \tilde{\gamma})^m, \; n \geq m \Rightarrow \delta_n = 0 \text{ for } n > m.$$

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Proposition

Take $\sigma(x) = \sigma(x, \gamma, m) = \frac{1}{(1-\gamma x)^m \sqrt{1-x^2}}$ with m a nonnegative integer and $\gamma \in (-1, 1)$ and fix $x_\alpha \in (-1, 1)$. Then, the quadrature rule $I_n^{r,s}(f)$ with $2n \geq m + r + s$ exists, if and only if,

$$(-1)^r \Im \left(z_\alpha^{2(m-n+(r+s))} \left(\frac{\overline{z}_\alpha - \tilde{\gamma}}{z_\alpha - \tilde{\gamma}} \right)^m \right) \leq 0,$$

where $z_\alpha = x_\alpha + i\sqrt{1 - x_\alpha^2}$ and $\tilde{\gamma} = 2(\frac{1}{\gamma} + \gamma)^{-1}$.

Remark

When taking above $m = 0$, then $\sigma(x) = \frac{1}{\sqrt{1-x^2}}$ and it follows

$$(-1)^r \sin(2(n - r - s)\alpha) \geq 0,$$

with $x_\alpha = \cos\alpha$ with $\alpha \in (0, \pi)$.

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Fix $m = 2$, consider $r = s = 0$ and take $n = 5$, $\sigma(x) = \frac{1}{(x-\gamma)^2\sqrt{1-x^2}}$, $\gamma \in (-1, 1)$.

γ	x_α	$\tilde{\delta}_{2n-1}$	$\tilde{\tau}_{2n}$
0.198	0.5403023059	0.8796981024	1
	-0.4161468365	-0.7176506326	-1
	0.2836621855	-0.3967315251	-1
	-0.1455000338	0.8576175107	-1
0.8	0.5403023059	0.2487954650	-1
	-0.4161468365	-0.3573679249	1
	0.2836621855	-0.4104362350	1
	-0.1455000338	-0.0236154680	-1
-0.975	0.5403023059	-0.5574209200	1
	-0.4161468365	0.4703245040	1
	0.2836621855	0.8612859631	1
	-0.1455000338	-0.9686333180	-1

Recall that $I_n^{r,s}(f)$ exists $\Leftrightarrow \tilde{\tau}_{2n-(r+s)} = (-1)^r$. Hence $I_n^{0,0}$ exists $\Leftrightarrow \tilde{\tau}_{2n} = 1$.